Lippmann-Schwinger's integral equation for quaternionic Dirac operators

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Abstract

We consider the scattering problem for the quaternionic Dirac operators $D+\nu$ and $D+M^{\boldsymbol{\alpha}}$, where ν is a scalar-valued function of x and $\boldsymbol{\alpha}$ is a quaternionic vector function depending on x. These operators are highly related to Maxwell's equations.

1 Introduction

In a lot of papers fundamental connections between electromagnetism and Clifford algebras are shown. We will restrict ourselves to the algebra of quaternions. In [6] and elsewhere especially Maxwell's equations are rewritten in terms of a Dirac operator $D+\alpha$. The advantage is that in this setting Maxwell's equations are treated as a system of first order differential equations.
To ensure the uniqueness of a non-homogeneous differential equation in the whole space additional conditions are needed. In case of the Helmholtz equation a radiation condition was introduced by A. Sommerfeld (see [8]). Scattering problems for the Helmholtz equation are considered in [1] and [2]. In [7] and [5] radiation conditions for $D+\nu$ (scalar-valued ν)and $D+\boldsymbol{\alpha}$ (vector-valued $\boldsymbol{\alpha}$) were given.
Based on these facts we consider an associated scattering problem for variable ν as well as $\boldsymbol{\alpha}$ and demonstrate the equivalence to a Lippmann-Schwinger integral equation.

2 The algebra of quaternions

Let $e_i,\ i=1,2,3$, be basic elements of the algebra of quaternions and e_0 the unit element of the algebra, i.e.

$$e_0^2 = e_0 = -e_j^2, \quad e_0e_j = e_je_0 = e_j, \quad j=1,2,3, \quad \text{(unit element)}$$
$$e_1e_2 = -e_2e_1 = e_3, \quad e_2e_3 = -e_3e_2 = -e_1, \quad e_3e_1 = -e_1e_3 = e_2 \quad \text{("clockwise")}$$

By i we denote the imaginary unit of complex numbers which commutes with all basis elements of the algebra of quaternions. Thus, an arbitrary element of the complex algebra of quaternions can be represented as

$$a = a_0e_0 + \sum_{j=1}^{3} a_je_j = \mathrm{Sc}\,a + \mathrm{Vec}\,a = a_0 + \mathbf{a},$$

where $\mathrm{Sc}\,a$ is called scalar part and $\mathrm{Vec}\,a$ is called vector part of a. The main anti-involution is given by

$$\bar{a} = a_0e_0 - \sum_{j=1}^{3} a_je_j = a_0 - \mathbf{a},$$

An element a is called zero divisor iff $\bar{a}a = a\bar{a} = 0$. If a is *not* a zero divisor then the inverse element a^{-1} is given by

$$a^{-1} = \frac{\bar{a}}{\bar{a}a}.$$

The vector part is essentially connected with vectors and their product. More precise, let $\mathbf{a}$, $\mathbf{b}$ vectors which are identified with vector parts of quaternions. We have

$$\mathbf{a} \cdot \mathbf{b} = \frac{1}{2} \left(\mathbf{ab} + \mathbf{ba} \right),$$

where $\mathbf{a} \cdot \mathbf{b}$ denotes the scalar product $\sum_{j=1}^{3} a_j b_j$, and

$$\mathbf{a} \times \mathbf{b} = \frac{1}{2} \left(\mathbf{ab} - \mathbf{ba} \right),$$

where $\mathbf{a} \times \mathbf{b}$ denotes the cross product

$$\mathbf{a} \times \mathbf{b} := \begin{vmatrix} e_1 & e_2 & e_3 \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}.$$

Let $G \subset R^3$ a bounded or unbounded domain. We identify $x = \sum_{j=1} x_j e_j = (x_1, x_2, x_3) \in \mathbb{R}^3$. The most important operator in quaternionic analysis is the so-called *Dirac operator*

$$D := \sum_{j=1}^{3} e_j \partial_j, \quad \text{where} \quad \partial_j = \frac{\partial}{\partial x_j}.$$

A complex quaternionic valued function is a combination of complex-valued functions and basic elements of the algebra of quaternions, more precise

$$f(x) = f_0(x) e_0 + \sum_{j=1}^{3} f_j(x) e_j.$$

The action of the operator D can be written in vector form

$$Df = -\operatorname{div} \mathbf{f} + \operatorname{grad} f_0 + \operatorname{curl} \mathbf{f}.$$

This is an immediate consequence of the quaternionic product. In a good number of physical applications the operators

$$D_\alpha = D + M^\alpha$$

are needed, where α is a complex quaternion and the multiplication operator M^α acts on functions f as a multiplication from the right-hand side $M^\alpha f = f\alpha$. In case of $\alpha = \operatorname{Sc} \alpha$ or $\alpha = \boldsymbol{\alpha}$ we have

$$-D_\alpha D_{-\alpha} = \Delta + \alpha^2 I,$$

where Δ denotes the Laplacian and I the identity operator.

3 Maxwell's equations and force free magnetic fields

In this section we give some examples of operators of Dirac-type. We want to write down Maxwell's equations for an inhomogeneous medium following [3]

$$\begin{aligned} \operatorname{curl} \vec{H} &= \varepsilon\, \partial_t\, \vec{E} + \vec{j}, \\ \operatorname{curl} \vec{E} &= -\mu\, \partial_t\, \vec{H}, \\ \operatorname{div} (\varepsilon\, \vec{E}) &= \rho, \\ \operatorname{div} (\mu\, \vec{H}) &= 0. \end{aligned}$$

Here $\varepsilon = \varepsilon(x)$ and $\mu = \mu(x)$ are absolute permittivity and permeability of the medium respectively. A very elegant quaternionic reformulation can be obtained by using the vectors

$$\mathbf{E} := \sqrt{\varepsilon}\,\mathbf{E} \quad \text{and} \quad \mathbf{H} := \sqrt{\mu}\,\mathbf{H},$$

which satisfy the following system

$$D_{\boldsymbol{\varepsilon}}\mathbf{E} = -\frac{1}{c}\,\partial_t\,\mathbf{H} - \frac{\rho}{\sqrt{\varepsilon}},$$
$$D_{\boldsymbol{\mu}}\mathbf{H} = \frac{1}{c}\,\partial_t\,\mathbf{E} + \sqrt{\mu}\,\mathbf{j},$$

where $c = 1/\sqrt{\varepsilon\mu}$ is the speed of propagation of electromagnetic waves in the medium;

$$\boldsymbol{\varepsilon} := \frac{\operatorname{grad}\sqrt{\varepsilon}}{\sqrt{\varepsilon}}, \quad \boldsymbol{\mu} := \frac{\operatorname{grad}\sqrt{\mu}}{\sqrt{\mu}}$$

and as before $D_{\boldsymbol{\varepsilon}} := D + M^{\boldsymbol{\varepsilon}}$, $D_{\boldsymbol{\mu}} := D + M^{\boldsymbol{\mu}}$. In a time-harmonic case with the dependence on the time as $\exp(i\omega t)$ the complex amplitudes of $\mathbf{E}$ and $\mathbf{H}$ which again will be denoted by $\mathbf{E}$ and $\mathbf{H}$ are solutions of the system

$$D_{\boldsymbol{\varepsilon}}\mathbf{E} = -i\alpha\,\mathbf{H} - \frac{\rho}{\sqrt{\varepsilon}},$$
$$D_{\boldsymbol{\mu}}\mathbf{H} = i\alpha\,\mathbf{E} + \sqrt{\mu}\,\mathbf{j},$$

where $\alpha := \omega\sqrt{\varepsilon\mu}$ is the wave number. There are two special cases which give raise to a decoupled system. The first situation is when the wave number α is a constant and we write down the so-called Beltrami fields:

$$\boldsymbol{\varphi} := \mathbf{E} + i\mathbf{H}, \qquad \boldsymbol{\psi} := \mathbf{E} - i\mathbf{H}.$$

Proposition 1 ([3]) *In case of a chiral homogeneous medium the diagonalized Maxwell's equations have the form*

$$D_{\alpha_1}\boldsymbol{\varphi} = 0 \quad \textit{and} \quad D_{\alpha_2}\boldsymbol{\psi} = 0$$

where $\alpha_1 := \alpha/(1+\beta\alpha)$ and $\alpha_2 := \alpha/(1-\beta\alpha)$ and β is the chirality measure of the medium.

Proposition 2 ([5]) *The second case are the static electric and magnetic fields in an inhomogeneous medium. Here, we have*

$$D_{\boldsymbol{\varepsilon}}\mathbf{E} = 0 \quad \textit{and} \quad D_{\boldsymbol{\mu}}\mathbf{H} = 0.$$

Proposition 3 ([4]) *Force free magnetic fields are an important special solution of non-linear equations of magnetohydrodynamics. They are characterized by the following pair of equations:*

$$\operatorname{div}\mathbf{B} = 0 \quad \textit{and} \quad \operatorname{curl}\mathbf{B} + \alpha\mathbf{B} = 0,$$

where $\alpha = \alpha(x)$ is a scalar-valued function. The system is equivalent to the quaternionic formulation

$$D_\alpha\mathbf{B} = 0.$$

4 Representation of solutions

We will need some properties of the fundamental solution of the Helmholtz equation.

Proposition 4 ([2]) *Let* $F_\nu(x-y) := -\dfrac{e^{i\nu|x-y|}}{4\pi|x-y|}$ *for* $x, y \in \mathbb{R}^3,\ x \neq y$.

$F_\nu(\cdot, y)$ *solves the Helmholtz equation* $\Delta u + \nu^2 u = 0$ *in* $\mathbb{R}^3\backslash\{y\}$ *for every* $y \in \mathbb{R}^3$ *and*

$$F_\nu(x-y) = -\frac{e^{i\nu|x|}}{4\pi|x|} e^{-i\nu\hat{x}\cdot y} + O(|x|^{-2})$$

$$\text{and} \quad D_x F_\nu(x-y) = (D_x F_\nu(x))\, e^{-i\nu\hat{x}\cdot y} + O(|x|^{-2})$$

uniformly in $\hat{x} = \frac{x}{|x|} \in S^2$ *and* $y \in G$ *for every bounded subset* $G \subset \mathbb{R}^3$.

If $\alpha = \mathrm{Sc}\,\alpha = \nu = const$ and $\mathrm{Im}\,\nu \geq 0$ the application of the operator $-D_{-\nu}$ to the fundamental solution of the Helmholtz operator $F_\nu(x)$ leads to

$$\mathcal{K}_\nu(x) = \left(\nu + \frac{x}{|x|^2} - i\nu\frac{x}{|x|}\right)\frac{e^{i\nu\,|x|}}{4\pi},\ x = \sum_{j=1}^{3} x_j e_j.$$

For more information see [6]. Let $\alpha = \boldsymbol{\alpha}$ be vector-valued quaternion.

Proposition 5 ([5]) *Let* G *be a bounded Lipschitz domain with boundary* Γ *and outward unit normal* $\mathbf{n}(y) = \sum_{j=1}^{3} n_j(y)e_j$. *Let* $f \in C^1(G) \cap C(G)$. *If* $f \in \ker D_\nu(G),\ \mathrm{Im}\,\nu \geq 0$ *then*

$$f(x) = K_\nu[f](x) = -\int_\Gamma \mathcal{K}_\nu(x-y)\,\boldsymbol{n}(y)\,f(y)\,d\Gamma_y.$$

If $f \in \ker D_{\boldsymbol{\alpha}} = \ker\,(D + M^{\boldsymbol{\alpha}})$ *then*

$$\begin{aligned} f(x) = K_{\boldsymbol{\alpha}}[f](x) &= -\int_\Gamma (-D_x F_\nu)(x-y)\boldsymbol{n}(y)f(y) + \boldsymbol{n}(y)f(y)F_\nu(x-y)\boldsymbol{\alpha}\,d\Gamma_y \\ &= -\int_\Gamma \left(\frac{x-y}{|x-y|^2} - i\nu\frac{(x-y)}{|x-y|}\right)\frac{e^{i\nu|x-y|}}{4\pi|x-y|}\boldsymbol{n}(y)\,f(y) \;-\; \frac{e^{i\nu|x-y|}}{4\pi|x-y|}\boldsymbol{n}(y)\,f(y)\,\boldsymbol{\alpha}\,d\Gamma_y, \end{aligned}$$

where $\nu = \sqrt{\boldsymbol{\alpha}^2} \in \mathbb{C}$ *is chosen such that* $\mathrm{Im}\,\nu \geq 0$.

Let us now consider the complement of G in $\mathbb{R}^3$, i.e. the unbounded domain $\mathbb{R}^3\backslash\bar{G}$.

Proposition 6 (Radiation condition ([5])) *Let* $f \in C^1(\mathbb{R}^3\backslash\bar{G}) \cap C(\mathbb{R}^3\backslash\bar{G})$, $f \in \ker D_\nu(\mathbb{R}^3\backslash\bar{G})$ *and* f *satisfies the* radiation condition

$$\left(\nu - \frac{x}{|x|^2} + i\nu\frac{x}{|x|}\right) f(x) = o(|x|^{-1}) \quad as \quad |x| \to \infty,$$

then

$$f(x) = K_\nu[f](x), \qquad \forall x \in \mathbb{R}^3\backslash\bar{G}.$$

If f *satisfies the* radiation condition

$$\nu\,f(x) + \frac{ix}{|x|}f(x)\boldsymbol{\alpha} = o(|x|^{-1}), \quad as \quad |x| \to \infty,$$

where $\nu := \sqrt{\boldsymbol{\alpha}^2} \in \mathbb{C}$ *and* $\mathrm{Im}\,\nu \geq 0$, *then*

$$f(x) = K_{\boldsymbol{\alpha}}[f](x), \quad for\ all \quad x \in \mathbb{R}^3\backslash\overline{G}.$$

If $\boldsymbol{\alpha}$ *is a zero divisor we suppose additional* $f(x) = o(1)$.
(Be aware that the outward pointed normal of $\mathbb{R}^3\backslash\bar{G}$ *is minus the outward pointed normal of* G.*)*

An immediate conclusion is

Proposition 7 (Uniqueness) *A function $f \in C^1(\mathbb{R}^3)$ that fulfils $Df + f\boldsymbol{\alpha} = 0$ in all of $\mathbb{R}^3$ and satisfies the radiation condition vanishes.*

Let $\Gamma_R = \partial B_R$ and $B_R := \{x \in \mathbb{R}^3 : |x| = R\}$. Due to Proposition 5 $f(x)$ can be represented as an integral over Γ_R and due to the properties of $F_\nu(x-y)$ we have $f(x) = o(1)$ as $|x| \to infty$. Now,

$$\begin{aligned}
f(x) = K_{\boldsymbol{\alpha}}[f](x) &= \int_{\Gamma_R} (-D_x F_\nu)(x-y)\boldsymbol{n}(y)f(y) + \boldsymbol{n}(y)f(y)F_\nu(x-y)\boldsymbol{\alpha}\, d\Gamma_{R,y} \\
&\sim \int_{\Gamma_R} (D_y F_\nu)(y)\frac{y}{|y|}f(y) + \frac{y}{|y|}f(y)F_\nu(y)\boldsymbol{\alpha}\, d\Gamma_{R,y} \\
&\sim \int_{\Gamma_R} F_\nu(y)\left\{\left(\frac{1}{|y|} - i\nu\right)f(y) + \frac{y}{|y|}f(y)\boldsymbol{\alpha}\right\} d\Gamma_{R,y} \\
&\sim \int_{\Gamma_R} F_\nu(y)\left\{\frac{1}{|y|} - i\left(\nu f(y) + i\frac{y}{|y|}f(y)\boldsymbol{\alpha}\right)\right\} d\Gamma_{R,y} \\
&\sim \int_{\Gamma_R} F_\nu(y)\left\{\frac{1}{|y|}f(y) + o(|y|^{-1})\right\} d\Gamma_{R,y} \to 0 \quad \text{as} \quad y \to \infty.
\end{aligned}$$

because f fulfils the radiation condition and $f(y) = o(1)$. Thus $f(x) = 0$.

5 The scattering problem

This section is devoted to the scattering problem for the Dirac operator $D + M^{\boldsymbol{\alpha}}$. The case of D_ν can be considered similarily. Statement of the problem: Let $1 - m \in C^1(\mathbb{R}^3)$ such that

$$1 - m$$

has compact support and let

$$G := \{x \in \mathbb{R}^3 : 1 - m(x) \neq 0\}.$$

Let $\boldsymbol{\alpha}$ be a vector-valued quaternion and $\nu = \sqrt{\boldsymbol{\alpha}^2}$ such that $\operatorname{Im}\nu \geq 0$. Further u^i is a solution of

$$Du^i(x) + u^i(x)\boldsymbol{\alpha} = 0 \quad \text{in} \quad \mathbb{R}^3.$$

Then the scattering problem consists in determining a scattering solution $u^s(x)$ such that

$$\begin{gathered}
Du + um(x)\boldsymbol{\alpha} = 0 \quad \text{in} \quad \mathbb{R}^3, \\
u = u^i + u^s,
\end{gathered}$$

and u^s fulfils the radiation condition

$$\nu\, u^s(x) + \frac{ix}{|x|}u^s(x)\boldsymbol{\alpha} = o(|x|^{-1}), \quad \text{as} \quad |x| \to \infty.$$

If $\boldsymbol{\alpha}$ is a zero divisor we additional assume $u^s(x) = o(1)$.

Theorem 1 *If u is a solution of the scattering problem then $u\,|_G$ solves the Lippmann-Schwinger integral equation*

$$u(x) = u^i(x) - \int_G (-D_x F_\nu(x-y)u(y)(\boldsymbol{\alpha} - m(y)\boldsymbol{\alpha}) + u(y)(\boldsymbol{\alpha} - m(y)\boldsymbol{\alpha})F_\nu(x-y)\boldsymbol{\alpha}\, dy. \quad (1)$$

Proof: First let u be a solution of the scattering problem and v the integral on the right-hand side for $x \in \mathbb{R}^3$. Since $u \in C^1(\mathbb{R}^3)$ we conclude from the properties of the volume potential that $v \in C^{1,\beta}(\mathbb{R}^3)$ and $Dv + v\boldsymbol{\alpha} = u(1-m)\boldsymbol{\alpha} = Du + u\boldsymbol{\alpha}$. Therefore, $w = u - v$ satiesfies $Dw + w\alpha = 0$ in all of $\mathbb{R}^3$. Furthermore,

$$w(x) = [u^i(x) - u(x)] - \int_G (-D_x F_\nu(x-y)u(y)(\boldsymbol{\alpha} - m(y)\boldsymbol{\alpha}) + u(y)(\boldsymbol{\alpha} - m(y)\boldsymbol{\alpha})F_\nu(x-y)\boldsymbol{\alpha}\, dy,$$

$x \in \mathbb{R}^3$, and satisfies the radiation condition which will be seen as follows:
$u^i(x) - u(x) = u^s(x)$ which satisfies the radiation condition by assumption that u is the solution of the scattering problem. Let us now conisder the integral. The dependence on x is given by $F_\nu(x-y)$ and $(-D_x F_\nu)(x-y)$ which can be written as

$$F_\nu(x-y) = F_\nu(x)e^{-i\nu\hat{x}\cdot y} + O(|x|^{-2}) \quad \text{and} \quad (D_x F_\nu)(x-y) = (D_x F_\nu)(x)e^{-i\nu\hat{x}\cdot y} + O(|x|^{-2}).$$

If we consider the expression under the integral we get

$$\begin{aligned}
&(-D_x F_\nu)(x-y)u(y)(1-m(y))\boldsymbol{\alpha} + \nu^2 u(y)(1-m(y))F_\nu(x-y) \\
&= \big((-D_x F_\nu)(x)u(y)(1-m(y))\boldsymbol{\alpha} + \nu^2 u(y)(1-m(y))F_\nu(x)\big)\, e^{-i\nu\hat{x}\cdot y} + O(|x|^{-2}).
\end{aligned}$$

Therefore it's enough to consider

$$(-D_x F_\nu)(x)u(y)(1-m(y))\boldsymbol{\alpha} + \nu^2 u(y)(1-m(y))F_\nu(x).$$

We have

$$\begin{aligned}
&\nu\left((-D_x F_\nu)(x)u(y)(1-m(y))\boldsymbol{\alpha} + \nu^2 u(y)(1-m(y))F_\nu(x)\right) \\
&\qquad\qquad + \frac{ix}{|x|}\left((-D_x F_\nu)(x)u(y)(1-m(y))\boldsymbol{\alpha} + \nu^2 u(y)(1-m(y))F_\nu(x)\right)\boldsymbol{\alpha} \\
&= \nu(-D_x F_\nu)(x)u(y)(1-m(y))\boldsymbol{\alpha} + \nu^3 u(y)(1-m(y))F_\nu(x) \\
&\qquad\qquad + \frac{ix}{|x|}(-D_x F_\nu)(x)u(y)(1-m(y))\boldsymbol{\alpha}^2 + \frac{ix}{|x|}\nu^2 u(y)(1-m(y))F_\nu(x)\boldsymbol{\alpha} \\
&= \left(\nu(-D_x F_\nu(x) + \frac{i\nu^2\, x}{|x|}F_\nu(x)\right)u(y)(1-m(y))\boldsymbol{\alpha} \\
&\qquad\qquad\qquad\qquad + \nu^2\left(\frac{ix}{|x|}(-D_x F_\nu)(x) + \nu F_\nu(x)\right)u(y)(1-m(y)) \\
&= -\left(\frac{\nu\, x}{|x|^3} - i\nu^2\frac{x}{|x|^2} + i\nu^2\frac{x}{|x|^2}\right)\frac{e^{i\nu\,|x|}}{4\pi}u(y)(1-m(y))\boldsymbol{\alpha} \\
&\qquad\qquad\qquad\qquad - \nu^2\left(\frac{ix^2}{|x|^4} + \nu\frac{x^2}{|x|^3} + \nu\frac{1}{|x|}\right)\frac{e^{i\nu\,|x|}}{4\pi}u(y)(1-m(y)) \\
&= -\frac{\nu\, x}{|x|^3}\frac{e^{i\nu\,|x|}}{4\pi}u(y)(1-m(y))\boldsymbol{\alpha} - \nu^2\frac{ix^2}{|x|^4}\frac{e^{i\nu\,|x|}}{4\pi}u(y)(1-m(y)) = O(|x|^{-2}).
\end{aligned}$$

Thus w fulfils the radiation condition and is a solution of $Dw + w\boldsymbol{\alpha}$ in all of $\mathbb{R}^3$. From the uniqueness theorem we get that $w = 0$ and hence $u = v$. This proves the first part.
To prove the second part let $u \in C(G)$ be a solution of 1. Extend u by the right-hand side of 1 to all of $\mathbb{R}^3$. From the properties of the volume potential and $\boldsymbol{\alpha} \in C^1(\mathbb{R}^3)$ we get $u \in C^{1,\beta}(\mathbb{R}^3)$. Furthermore, $Du + u\boldsymbol{\alpha} = u(1-m(y))\boldsymbol{\alpha}$ and thus $Du + u\, m(y)\,\boldsymbol{\alpha} = 0$ in $\mathbb{R}^3$. The same observations as in the previous part demonstrate that the scattered field $u^s = u - u^i$ fulfils the radiation condition.

6 Remarks

- It is not difficult to prove an anlog of Theorem 1 for $D + \nu(x)$.
- It can also be proven that the scattering problem is uniquely solvable. For that a unique continuation principle is needed.
- We can use the unique continuation principle for the Helmholtz equation but this automatically involves a higher differentiability of u and the existence of first order derivaties of ν and $\boldsymbol{\alpha}$ respectively. Therefore, we are interested in a unique continuation principle for $D + \nu$ and $D + \boldsymbol{\alpha}$.
- In an analogous way to the scattering problem for the Helmholtz equation a far field pattern can be defined.

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